SECTION 17.3: CONSERVATIVE VECTOR FIELDS

RECALL: Given a function $\phi(x, y)$, the gradient $\nabla\phi(x, y)$ is a vector field!

QUESTION: Are all vector fields gradients?

EXAMPLE 1: Consider $\vec{F}(x, y) = \langle 2y, -x \rangle$. Try to find a function $\phi(x, y)$ so that $\nabla\phi(x, y) = \vec{F}(x, y)$.

HINT: If $\nabla\phi(x, y) = \vec{F}(x, y)$, then $\phi_x(x, y) = 2y$ and $\phi_y(x, y) = -x \ldots$

RECALL: If there is a function ϕ so that $\nabla\phi = \vec{F}$, then $\vec{F}$ is called a **gradient (or conservative) field**.

NOTE: The function ϕ is called a **potential** function for the field $\vec{F}$.

How do we determine if a field is conservative? We use the following theorem.

THEOREM: Suppose $\vec{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ where M and N have continuous first partial derivatives.

Then $\vec{F}$ is conservative **if and only if** $M_y(x, y) = N_x(x, y)$.

NOTE: The equality $M_y(x, y) = N_x(x, y)$ is called the **exactness criterion**

EXAMPLE 2: Let $\vec{F}(x, y) = \left\langle e^{xy} + xy\, e^{xy}, x^2 e^{xy} \right\rangle$.

1. Use the exactness criteria to show $\vec{F}$ is conservative.

Ans: $M_y(x, y) = 2xe^{xy} + x^2ye^{xy} = N_x(x, y)$ ✓

2. Find a potential function ϕ for $\vec{F}$.

Ans: $\phi(x, y) = xe^{xy} + C$; $\nabla\phi(x, y) = \vec{F}(x, y)$ ✓

EXTENSIONS TO THREE DIMENSIONS:

If $\vec{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ where M, N, and P have continuous first partial derivatives:

$\vec{F}$ is conservative if and only if $M_y(x, y, z) = N_x(x, y, z)$, $M_z(x, y, z) = P_x(x, y, z)$, and $N_z(x, y, z) = P_y(x, y, z)$.

EXAMPLE 3: Let $\vec{F}(x, y, z) = \langle z, -z\cos(yz), x - y\cos(yz)\rangle$.

1. Use the exactness criterion to prove $\vec{F}$ is conservative.

Ans: $M_y(x, y, z) = 0 = N_x(x, y, z)$, $M_z(x, y, z) = 1 = P_x(x, y, z)$,
and $N_z(x, y, z) = -\cos(yz) + yz\sin(yz) = P_y(x, y, z)$ ✓

2. Find a potential ϕ for $\vec{F}$.

Ans: $\phi(x, y, z) = xz - \sin(yz) + C$; $\nabla\phi(x, y, z) = \vec{F}(x, y, z)$ ✓

THEOREM: THE FUNDAMENTAL THEOREM OF LINE INTEGRALS

Let R be an open region in the plane containing a piecewise smooth oriented curve C with initial point (a, b) and terminal point (c, d). Let $\vec{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ be a conservative vector field on R where $M(x, y)$ and $N(x, y)$ are continuous. Let $\phi(x, y)$ be any potential function for $\vec{F}(x, y)$ on R. Then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla\phi \cdot d\vec{r} = \phi(c, d) - \phi(a, b).$$

PROOF:

NOTE: In particular, this theorem says that the **ONLY** thing important to know about C is where it **STARTS** and where it **ENDS**! In other words, for conservative vector fields, it doesn't matter which path you choose to get from (a, b) to (c, d) because

$$\int_C \vec{F} \cdot d\vec{r} = \phi(c, d) - \phi(a, b)$$

which depends on $\vec{F}$ but not on the path C which connects (a, b) to (c, d).

In this case, we say the integral $\int_C \vec{F} \cdot d\vec{r}$ is **path independent**.

Hence, if $\vec{F}$ is conservative, to find the value of $\int_C \vec{F} \cdot d\vec{r}$ you have options:

- You can evaluate the integral along C as indicated.
- You can replace C with a simpler path from (a, b) to (c, d) (i.e., a line segment).
- You can find a potential ϕ for $\vec{F}$ and use the Fundamental Theorem of Line Integrals.

EXAMPLE 4: Let $\vec{F}(x,y) = \langle 2xy, x^2 \rangle$.

1. Show $\vec{F}$ is conservative and find a potential ϕ for $\vec{F}$.

Ans: $\phi(x,y) = x^2y + C$.

2. Find $\int_C \vec{F} \cdot d\vec{r}$ where:

   (a) C is the directed line segment from $(0,1)$ and ending at $(1,4)$.

   Ans: $\phi(1,4) - \phi(0,1) = 4$.

   (b) C is the curve starting at $(1,4)$ and ending at $(0,1)$ along the curve $y = 3x^2 + 1$.

   Ans: $\phi(0,1) - \phi(1,4) = -4$.

   (c) C is a two-part path: the first part is the path in part (i) the second in part (ii)

   Ans: $\phi(0,1) - \phi(0,1) = 0$.

NOTE: The curve C in the last example ends where it begins. Such a curve is called a **closed** curve.

If C is closed, the notion $\oint_C \vec{F} \cdot d\vec{r}$ is often used.

QUESTION: If $\vec{F}$ and C satisfy the conditions of the Fundamental Theorem of Line Integrals what is $\oint_C \vec{F} \cdot d\vec{r}$?

Ans: $\oint_C \vec{F} \cdot d\vec{r} = 0$.

EXAMPLE 5: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = \langle \cos(\pi y), -\pi x \sin(\pi y) \rangle$ and C is the three-part path:

1. beginning at $(0, 0)$ along the line $y = 3x$ to the point $(2, 6)$,
2. then from $(2, 6)$ along the line $y = 8 - x$ to the point $(8, 0)$,
3. and then along the semicircle $y = -\sqrt{64 - x^2}$ back to $(-8, 0)$.

HINT: Check to see if $\vec{F}$ is conservative ...

Ans: $\phi(x, y) = x \cos(\pi y) + C$ is a potential function so $\int_C \vec{F} \cdot d\vec{r} = \phi(-8, 0) - \phi(0, 0) = -8$

The Fundamental Theorem for Line Integrals also holds in three space:

EXAMPLE 6: Let $\vec{F}(x, y, z) =< -2xy, e^{3z} - x^2, 3ye^{3z} >$

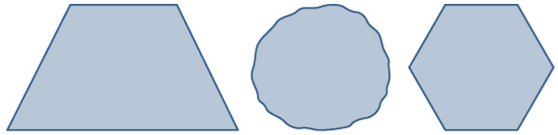
1. Show $\vec{F}$ is conservative.

   Ans: $M_y(x, y, z) = -2x = N_x(x, y, z)$, $M_z(x, y, z) = 0 = P_x(x, y, z)$, $N_z(x, y, z) = 3e^{3z} = P_y(x, y, z)$ ✓

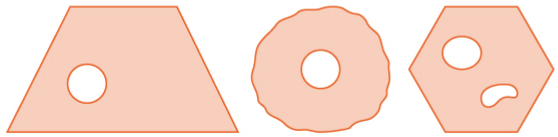
2. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where C is any curve starting at $(1, -3, 0)$ to $(2, -1, \ln(2))$.

   Ans: $\phi(x, y, z) = ye^{3z} - x^2y + C$ is a potential so $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \phi(2, -1, \ln(2)) - \phi(1, -3, 0) = -4$

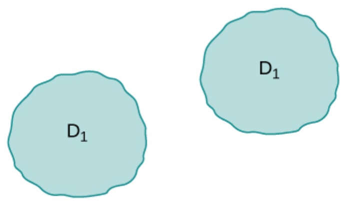
SOME VOCABULARY:



(a) Simply connected regions



(b) Connected regions that are not simply connected



(c) A region that is not connected

SUMMARY OF RESULTS:

THEOREM: Suppose $\vec{F}$ is a vector field with continuous component functions in an open connected region R.

Then the following are equivalent: (this means if one is true, all are true; if one is false, all are false.)

- $\vec{F}$ is conservative.
- $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \phi(\text{end point}) - \phi(\text{starting point})$ for any potential ϕ of $\vec{F}$.
- $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is independent of path.
- $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths C.

If, in addition, R is **simply** connected, then we may add the following:

- The components of $\vec{F}$ satisfy the exactness criteria:
  - (2D): If $\vec{F} = \langle M, N \rangle$ then $M_y = N_x$
  - (3D): If $\vec{F} = \langle M, N, P \rangle$ then $M_y = N_x$, $M_z = P_x$ and $N_z = P_y$.

EXAMPLE 7: Let $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$.

1. Show $M_y(x,y) = N_x(x,y)$

Ans: $M_y(x,y) = \dfrac{y^2-x^2}{(x^2+y^2)^2} = N_x(x,y)$

2. Let C be the Unit Circle, oriented counter-clockwise. Show $\oint_C \vec{F}\cdot d\vec{r} \neq 0$.

   HINT: Parametrize the Unit Circle as: $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq 2\pi$.

Ans: $\oint_C \vec{F}\cdot d\vec{r} = ... = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle \; dt = 2\pi \neq 0$

3. Is $\vec{F}$ conservative? Explain.

Ans: No, $\vec{F}$ is not conservative since $\oint_C \vec{F}\cdot d\vec{r} \neq 0$.

HOMEWORK: Section 17.3: 9 - 65 every other odd; 67, 71*

BONUS TRACKS: What do conservative fields conserve?

Suppose an object of mass m traverses a smooth curve C parameterized by $\vec{r}(t)$ through a continuous field $\vec{F}$.

- The **kinetic energy** of an object moving along C is given by: $k(t) = \frac{1}{2}m\,\|\vec{v}(t)\|^2$.

  Rewriting $k(t) = \frac{1}{2}m\,\|\vec{v}(t)\|^2 = \frac{1}{2}m\,\vec{v}(t)\cdot\vec{v}(t)$, use properties of derivatives to show: $k\,'(t) = m\,\vec{a}(t)\cdot\vec{v}(t)$.

- If C is parametrized over the interval $a \le t \le b$, then using Newton's Second Law, we get:

$$\begin{aligned} W &= \int_C \vec{F}\cdot d\vec{r} \\ &= \int_a^b (m\vec{a}(t))\cdot\vec{r}\,'(t)\,dt \\ &= \int_a^b m\,\vec{a}(t)\cdot\vec{v}(t)\,dt \\ &= \int_a^b k'(t)\,dt \\ W &= k(b) - k(a) \end{aligned}$$

  This shows that the work done moving an object through a continuous vector field is the difference in kinetic energy between the starting and ending points.

- If $\vec{F}$ is **conservative** with potential ϕ then we define the **potential energy** $U(t) = -\phi(t)$.

  In this case the Fundamental Theorem of Line Integrals gives:

$$W = \int_C \vec{F}\cdot d\vec{r} = \phi(b) - \phi(a) = -U(b) + U(a) = U(a) - U(b)$$

  This shows that the work done moving an object through a conservative field is the difference in potential energy between the starting and ending points.

- Hence for conservative fields: $k(b) - k(a) = U(a) - U(b)$ or $U(a) + k(a) = U(b) + k(b)$.

  This says that the **total energy** is the same at the starting and ending points. Said differently,

CONSERVATIVE FIELDS CONSERVE ENERGY !!!